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LETTER TO THE EDITOR

Finite-range scaling for a one-dimensional system with long-range interactions

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Abstract. A procedure, similar to the finite-size scaling, is proposed for systems with long-range interactions. It is applied to the one-dimensional Ising model with interactions decaying as $1/|i-j|^{1+\sigma}$ with distance $|i-j|$. The critical temperature and exponent ν are calculated for $\sigma \leqslant 1$.

In recent years, finite-size scaling (FSS) has been widely used as an efficient technique among other renormalisation group (RG) approaches, especially in the analysis of critical behaviour in low-dimensional systems (for a review see Barber (1983)). Its advantages in comparison to the direct space and other RG methods have been its better convergence and its easier tractability, especially for different problems with discrete symmetries. A class of problems difficult to handle within the direct space RG are systems with long-range interactions. Due to the non-local character of the interactions, these systems have been studied principally within momentum space through the ε expansions around upper (Fisher *et al* 1972) and lower (Brézin *et al* 1976, Kosterlitz 1976, Bulgadaev 1984) critical dimensions, which correspond to the $4-\varepsilon$ and $2+\varepsilon$ expansions for systems with short-range interactions. However, the intermediate region, which for short-range interaction systems has been studied using a number of direct space RG techniques including FSS, could not be explored to the same extent there. This letter is an attempt to give a new and more direct approach to this region by using the long-range interaction property as a basic scaling parameter in order to construct a scaling procedure similar to the FSS.

We take as an example the one-dimensional Ising model with power law decaying interactions represented by the Hamiltonian

$$H = -\sum_{i,j} J_{|i-j|} s_i s_j \tag{1}$$

where i and j denote sites on a chain and $J_{|i-j|} = J_0/|i-j|^{1+\sigma}$ is the interaction between $|i-j|$th neighbours. For $0<\sigma\leqslant 1$ this system has a phase transition at finite T_c (Dyson 1969), which is of mean field type for $\sigma<0.5$ and non-classical for $0.5<\sigma\leqslant 1$. The point $\sigma=1$ is analogous to the case $d=n=2$ of the n-component short-range interaction system. It divides the region $T_c \neq 0$ from $T_c=0$. The transition is mediated by topological defects and has an essential singularity instead of power law critical behaviour. For $\sigma>1$, $T_c=0$ and short-range behaviour takes place (Kosterlitz 1976). Critical behaviour in the region $0.5<\sigma\leqslant 1$ has already been examined by different approximate methods including the extrapolations of numerical results for finite chains (Nagle and Bonner 1970) and two different RG approaches. One is an ε expansion in σ around the mean field edge $\sigma = 0.5$ by Fisher *et al* (1972). The other is the expansion around $\sigma = 1$ (Kosterlitz 1976) following the method of Polyakov (1975).

In the present approach the intention is to use the exact results which can be obtained for an infinite chain with finite range of interactions N defined as

$$J_n = J_0 \times \begin{cases} n^{-(1+\sigma)} & \text{for } n \leq N \\ 0 & \text{for } n > N. \end{cases} \tag{2}$$

The basic idea is then to establish the scaling relations between two such systems with different finite ranges and relate them to the critical properties of the system with infinite range. The procedure is analogous to that of FSS and relies on a similar assumption. Let $C(t)$ be some critical quantity diverging at T_c ($t = (T - T_c)/T_c$). Then we assume that the finite range of interaction will prevent this divergence and produce a correction depending on the ratio of the range and correlation length of the infinite-range system N/ξ_∞, i.e.

$$C_N(t) = C_\infty(t) f(N/\xi_\infty). \tag{3}$$

Applying (3) to two systems with different ranges N and M it is straightforward to obtain the scaling relation for the correlation length

$$\xi_N(t) = (N/M)\xi_M[(N/M)^{1/\nu} t] \tag{4}$$

where the critical temperature and critical exponent are calculated in a standard way and are given by the expressions

$$\xi_N(0)/\xi_M(0) = N/M \tag{5}$$

$$1/\nu = \frac{\ln[\xi'_N(0)/\xi'_M(0)]}{\ln(N/M)} - 1. \tag{6}$$

For Hamiltonian (1) ξ_N and ξ_M can be calculated by a transfer matrix method. As illustrated in figure 1, grouping the spins by N and changing to new, N-component variables α_j transforms the problem into a nearest-neighbour problem with 2^N degrees of freedom per site. The components $\alpha_j(i)$ adopt the values ± 1 corresponding to the ith spin of the jth column in figure 1. The transfer matrix T is then expressed by

$$T_{j,j+1} = \exp(-\{\beta[H_{j,j+1} + (H_j + H_{j+1})/2]\}) \tag{7}$$

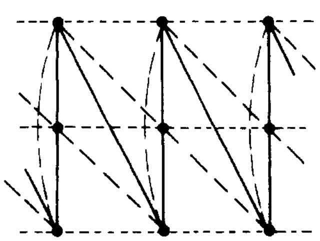

Figure 1. Chain for the case $N = 3$, drawn in zig-zag. Interactions J_1, J_2, J_3 are represented by full, long broken and short broken lines respectively. α_j is a three-component variable describing all configurations of 3-spins in column j.

where

$$H_{j,j+1} = -\sum_{m=0}^{N-1} J_{N-m} \sum_{i=1}^{N-m} \alpha_j(i)\alpha_{j+1}(i+m)$$
$$H_j = -\sum_{m=1}^{N-1} J_m \sum_{i=1}^{N-m} \alpha_j(i)\alpha_j(i+m). \tag{8}$$

The correlation length is given by

$$\xi_N = N/\ln(\Lambda_1/\Lambda_2) \tag{9}$$

where Λ_1 and Λ_2 are the two largest eigenvalues of T. The matrix T is not symmetric. Originally of order 2^N, it can be reduced by a factor of two.

The numerical calculations presented here have been performed up to the range $N=8$. Scaling between ranges $N-1$ and N has been considered. The results for critical temperature and critical exponent ν in the region $T_c \neq 0$ are displayed for different values of σ from 0.1 to 1 as a function of N in tables 1 and 2 respectively.

Table 1. Results for critical temperature T_c as a function of scaled ranges (M, N) and parameter σ. The last column shows the extrapolated values.

σ	(3, 4)	(4, 5)	(5, 6)	(6, 7)	(7, 8)	extr.
0.1	6.9774	8.0396	8.9123	9.6500	10.2868	18.759
0.2	5.6118	6.2772	6.7883	7.1951	7.5277	10.552
0.3	4.6294	5.0679	5.3888	5.6338	5.8271	7.270
0.4	3.8912	4.1900	4.4010	4.5577	4.6783	5.483
0.5	3.3168	3.5240	3.6665	3.7702	3.8487	4.339
0.6	2.8568	3.0007	3.0975	3.1668	3.2187	3.530
0.7	2.4797	2.5774	2.6417	2.6871	2.7207	2.916
0.8	2.1643	2.2267	2.2665	2.2939	2.3139	2.421
0.9	1.8959	1.9301	1.9502	1.9631	1.9719	2.008
0.95	1.7761	1.7981	1.8095	1.8159	1.8196	1.828
0.98	1.7082	1.7233	1.7298	1.7324	1.7332	1.734
1.00	1.6645	1.6751	1.6784	1.6786	1.6774	1.677

Table 2. Results for critical exponent ν as a function of scaled ranges (M, N) and parameter σ. The last column shows the extrapolated values.

σ	(3, 4)	(4, 5)	(5, 6)	(6, 7)	(7, 8)	extr.
0.1	2.770	3.021	3.258	3.479	3.686	8.71
0.2	2.500	2.654	2.792	2.916	3.027	4.81
0.3	2.312	2.405	2.486	2.555	2.615	3.39
0.4	2.185	2.241	2.286	2.324	2.356	2.70
0.5	2.108	2.139	2.164	2.184	2.200	2.35
0.6	2.071	2.090	2.103	2.113	2.121	2.17
0.7	2.073	2.088	2.097	2.104	2.109	2.13
0.8	2.110	2.131	2.146	2.156	2.164	2.22
0.9	2.186	2.225	2.254	2.278	2.297	2.61
0.95	2.239	2.292	2.334	2.369	2.399	3.40
0.98	2.276	2.339	2.391	2.435	2.473	4.90
1.00	2.302	2.374	2.433	2.483	2.528	8.06

For $\sigma < 1$ both series of results show a monotonic dependence on N, while their convergence varies considerably with σ. This convergence deserves more detailed investigation in the future, for comparison with studies existing for the FSS (Privman and Fisher 1983), as well as in order to improve our results quantitatively. At present, we limit ourselves to reporting (in the last columns of tables 1 and 2) the results obtained by extrapolation to $N = \infty$ using the asymptotic corrections. The correction of the form $(1/N)^x$ was applied to T_c and to $y = \nu^{-1}$ for $\sigma > 0.5$. In the mean field region, however, the values of $\nu_{M,N}$ fit better to the form $(M/N)^x$. The parameter x and proportionality constant are adjustable and depend on σ, so that the error of the extrapolations themselves also depends on σ, but it was estimated to be a few per cent for the whole region considered.

Figures 2 and 3 represent comparisons of these extrapolated results with those obtained by other methods.

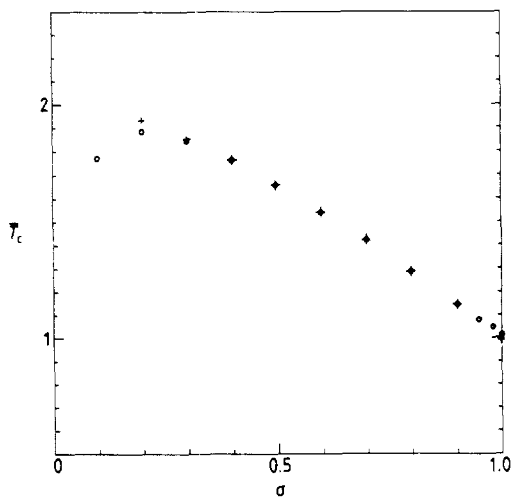

Figure 2. Extrapolated results for the critical temperature normalised by zero-temperature energy: $\bar{T}_c = T_c/[J_0\zeta(1+\sigma)]$ (open circles) compared to the results of Nagle and Bonner (1970) (crosses).

Results for critical temperature (figure 2) are compared with those available from Nagle and Bonner (1970) and show good agreement.

As should be expected, a comparison of different approaches shows more diversity for the critical exponent ν (figure 3). The discrepancy of our results is largest around $\sigma = 0.5$, which can be attributed (as for the results of Nagle and Bonner) to the neglect of logarithmic corrections. In the mean field region the deviations from the exact value $1/\sigma$ grow with decreasing σ. This could be related to the fact that our truncation has a stronger effect when σ is decreased, which slows down the convergence. In the non-trivial region our values are systematically higher than those of Nagle and Bonner. Few additional points taken close to $\sigma = 1$ show good agreement with the $\sigma = 1 - \varepsilon$ expansion. However, the essential singularity in $\sigma = 1$ is not reproduced at the present stage but manifests itself through a large exponent ν.

The results presented here cover only the $T_c \neq 0$ region, i.e. $\sigma \leq 1$, where long-range interactions are relevant. When applying the same procedure to $\sigma > 1$, a changeover

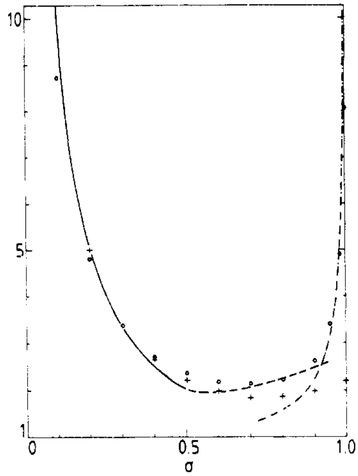

Figure 3. Extrapolated results for ν (open circles), compared to the exact results (full curve), Fisher *et al* (1972) (broken curve), Kosterlitz (1976) (chain curve) and Nagle and Bonner (1970) (crosses).

in the regime is observed and shows up in two ways: $(T_c)_{M,N}$ changes into a decreasing function of N, while ν becomes diverging with N.

In conclusion, we have shown that the proposed procedure gives results for arbitrary σ which are qualitatively good and quantitatively comparable with those obtained by other approximate methods. Further studies of convergence are currently under consideration. We expect that this method could be a useful tool for other problems involving long-range interactions. To conclude, let us mention one fault in comparison with usual FSS: the lack of a beautiful interpretation through conformal invariance theory.

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